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Morton E. Gurtin[1], Victor J. Mizel[1], and David W. Reynolds[2]

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ABSTRACT

In this note we consider the linearized equation for the buckling of a viscoelastic rod from an undeformed virgin state. We show that this equation does not exhibit buckled solutions for axial end thrusts which - after application - are held constant. Though this result is apparently known, there appears to be no proof available in the literature.

We show further that if the load is allowed to vary with time, then, in contrast to elastica theory, there is an uncountably infinite number of buckled solutions.

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SIGNIFICANCE AND EXPLANATION

The recent use of polymeric materials for structural purposes renders important the careful study of viscoelastic buckling, a phenomenon quite different from that associated with elastic materials, especially when the time scale of interest is large. In this paper we show that the linearized equation for the buckling of a viscoelastic rod from its virgin state does not exhibit buckled solutions for axial end thrusts which - after application - are held constant. We show further that if the load is allowed to vary with time, then, in contrast to elastica theory, there is an uncountably infinite number of buckled solutions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE BUCKLING OF LINEAR VISCOELASTIC RODS

Morton E. Gurtin[1], Victor J. Mizel[1], and David W. Reynolds[2]

1. Basic equations.

Consider a thin inextensible rod, pinned[3] at the ends, and acted on by an axial end thrust $P(t)$ in such a way that its center-line bends in a plane. For each material point x and time t, let $\varphi(x,t)$ denote the angle between the horizontal axis and the tangent to the rod at x (Figure 1), and let $m(x,t)$ designate the bending moment at x. Here, for convenience, we label material points by their positions $x \in [0,L]$ in the undeformed, straight configuration the rod is assumed to be in prior to time zero.

We work within the *quasi-static* theory; thus balance of moments has the form[4]

$$m' + P \sin \varphi = 0, \tag{1}$$

where $m' = \partial m/\partial x$. We assume that the rod is linearly viscoelastic in the sense of the constitutive equation[5]

$$m(x,t) = \beta\varphi'(x,t) + \int_0^t \alpha(t-s)\varphi'(x,s)\,ds \tag{2}$$

giving the bending moment as a function of the past history of the curvature φ'. Here $\beta > 0$ is the instantaneous flexural

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[3] In the interest of brevity, we consider only pinned ends. Our results go through, without change, for any of the standard boundary conditions. (In this connection, cf. [1].)

[4] Cf., e.g., [2], §262.

[5] Here we use the fact that $\varphi(x,t) = 0$ for $t < 0$.

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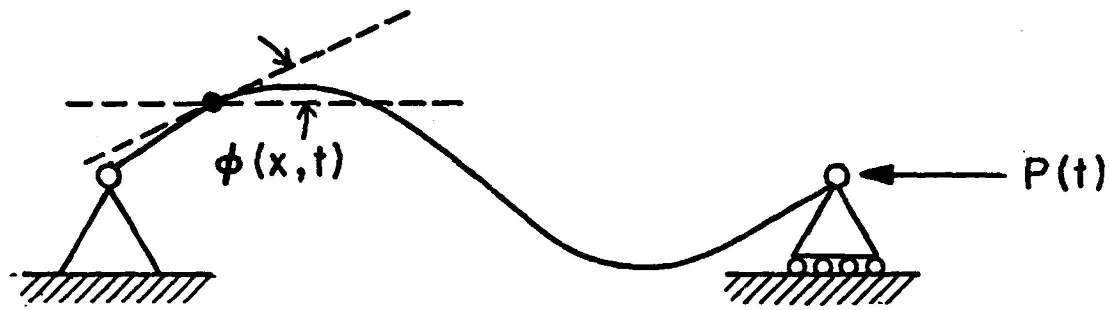

Figure 1. The deformed rod

rigidity, while α, assumed continuous and <0, measures internal dissipation; the unique solution $G(s)$, $s \geq 0$, of the initial-value problem

$$\dot{G}(s) = \alpha(s), \quad G(0) = \beta$$

is the (moment-curvature) relaxation function.

Equations (1) and (2) yield the integro-differential equation

$$\beta\varphi''(x,t) + \int_0^t \alpha(t-s)\varphi''(x,s)ds + P(t)\sin\varphi(x,t) = 0,$$

or equivalently, using the standard notation for *convolutions*,

$$\beta\varphi'' + \alpha * \varphi'' + P \sin\varphi = 0.$$

Since the ends of the rod are pinned, the relevant boundary conditions are

$$m(0,t) = m(L,t) = 0$$

for all t, or, by (2) and the standard uniqueness theorem for linear Volterra integral equations,

$$\varphi'(0,t) = \varphi'(L,t) = 0$$

for all t. Further, as the end of the rod corresponding to the material point $x = L$ can move only horizontally, we have the additional constraint

$$\int_0^L \sin \varphi(x,t)dx = 0.$$

In this note we confine our attention to the linear theory and hence replace $\sin \varphi$ by φ in the above equations. The resulting boundary-value problem then takes the form[1]

$$\begin{aligned} &\beta\varphi'' + \alpha * \varphi'' + P\varphi = 0, \\ &\varphi'(0,t) = \varphi'(L,t) = 0, \qquad (3) \\ &\int_0^L \varphi(x,t)dx = 0. \end{aligned}$$

[1] Cf. [3],[4] for a discussion of equations similar to $(3)_1$, but with P = constant. To our knowledge no previous paper has allowed $P(t)$ to depend on t.

2. Impossibility of buckled solutions when $P(t)$ = constant.

In this section we show that the problem (3) has only the zero solution when

$$P(t) = P = \text{constant for } t \geq 0.$$

With this in mind, let $0 < P_1 < P_2 < \ldots < P_n < \ldots$ and $\psi_1(x), \psi_2(x), \ldots, \psi_n(x), \ldots$ denote the eigenvalues and eigenvectors of the corresponding elastic problem:

$$\begin{aligned} &\beta\psi_n'' + P_n\psi_n = 0, \\ &\psi_n'(0) = \psi_n'(L) = 0, \\ &\int_0^L \psi_n(x)\,dx = 0; \end{aligned} \tag{4}$$

i.e.,

$$\psi_n(x) = \cos\frac{n\pi x}{L}, \quad P_n = \beta\,\frac{n^2\pi^2}{L^2}, \quad n = 1,2,\ldots \tag{5}$$

Suppose that $\varphi(x,t)$ is a sufficiently smooth solution of (3) and define

$$\varphi_n(t) = \int_0^L \varphi(x,t)\psi_n(x)\,dx. \tag{6}$$

Then, since φ and ψ_n have vanishing spatial derivatives at the ends of the rod, two integrations by parts in conjunction with $(4)_1$ yield

$$\int_0^L \varphi''\psi_n dx = \int_0^L \varphi\psi_n'' dx = -\frac{P_n}{\beta}\varphi_n .$$

Thus multiplying $(3)_1$ by ψ_n and integrating with respect to x over the interval $(0,L)$ results in the integral equation

$$(P-P_n)\varphi_n - \frac{P_n}{\beta}\alpha * \varphi_n = 0 \tag{7}$$

for $n = 1,2\ldots.$

If $P \neq P_n$, (7) is a Volterra equation of the second kind, which has the unique solution

$$\varphi_n(t) = 0 \tag{8}$$

for all t. On the other hand, if $P = P_n$,

$$\alpha * \varphi_n = 0,$$

and hence, by Titchmarsh's theorem[1], (8) holds in this case as well. By (6) and (8), φ is orthogonal to each function ψ_n of the form $(5)_1$. This is clearly possible only if $\varphi(x,t)$ is independent of x, and the desired conclusion, $\varphi \equiv 0$, follows from $(3)_3$.

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[1] Cf., e.g., [5], Theorem 152.

3. Existence of buckled solutions with $P(t) \neq$ constant.

There are buckled solutions for certain nonconstant loads. To see this consider solutions of the form

$$\varphi(x,t) = \gamma(t)\psi_n(x)$$

for $t \geq 0$. By $(4)_{2,3}$ this function satisfies the boundary conditions $(3)_2$ and the constraint condition $(3)_3$. Moreover, by $(4)_1$, φ will satisfy the differential equation $(3)_1$ provided γ and P satisfy the integral equation

$$[P(t) - P_n]\gamma(t) - \frac{P_n}{\beta}\int_0^t \alpha(t-s)\gamma(s)ds = 0. \tag{9}$$

This equation can have many solutions. A simple example is

$$\gamma(t) = \text{constant}, \quad P(t) = P_n\mathcal{G}(t),$$

with

$$\mathcal{G}(t) = \frac{G(t)}{G(0)}.$$

For this solution the angle $\varphi(x,t)$ jumps to its elastic shape at $t = 0$ and remains there for $t > 0$. The requisite axial thrust is initially the elastic buckling load P_n, but for $t > 0$, $P(t)$ decreases in proportion to the relaxation function $G(t)$.

Solutions continuous in time at $t = 0$ are also possible. For example,

$$\gamma(t) = At \quad (A = \text{constant}), \qquad P(t) = P_n \frac{1}{t} \int_0^t G(s)ds,$$

forms a solution. (Note that, again, $P(0^+) = P_n$.)

More generally, (9) can be rewritten as

$$\frac{P(t)}{P_n} = \frac{\gamma(t) + \gamma * \dot{G}(t)}{\gamma(t)}; \tag{10}$$

thus _any_ function $\gamma > 0$ on $[0,\infty)$ yields a buckled solution with P given by (10). It is not difficult to show that $P(0^+)$ exists when $\gamma(0^+) = 0$, provided $\dot{\gamma}(0^+) > 0$. Thus any smooth function γ on $[0,\infty)$ with $\gamma > 0$ on $(0,\infty)$, $\gamma(0) = 0$, and $\dot{\gamma}(0^+) > 0$ also generates a buckled solution.[1]

Thus, interestingly, in contrast to elastica theory there is an uncountably infinite number of buckled solutions (even modulo multiplicative constants).

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[1]Conditions on $P(t)$ which insure the existence of nontrivial $\gamma(t)$ are given in [1]. This paper also investigates the asymptotic behavior of buckled solutions.

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